On the Damping Term in the Polar Motion Equation

M.Furuya
CIRES and Department of Physics
University of Colorado, Boulder, CO 80309–390, USA
(Earthquake Research Institute, University of Tokyo, Yayoi 1–1–1, Bunkyo-ku, Tokyo 113-0032, JAPAN)

Abstract. In the conventional polar motion equation, two different usages of imaginary unit i have been employed. From the view point that the i should only be employed for a numerical convenience, we re-derive the polar motion equation with a particular attention to the damping term. We show that two kinds of damping terms are allowed for the polar motion equation.

When the damping term is proportional to the wobble amplitude, we need to multiply the standard excitation term by a correction factor so that we can exactly derive the "observed" excitation. Depending on the Q-value, this factor will potentially cause measurable changes in the magnitude and phase of the excitation, and possible biases into the estimation of Chandler period and Q.
Keywords. Damping, Imaginary unit, Chandler wobble, Polar motion

1 Introduction

For the governing equation of the Earth's wobble, the following equation has been conventionally used:

$$\frac{i}{\tilde{\sigma}_{CW}}\frac{d\tilde{\mathbf{m}}}{dt} + \tilde{\mathbf{m}} = \tilde{\chi}, \quad (1)$$

where $\tilde{\mathbf{m}}$ stands for wobble, and is defined as $m_1 + im_2$ when the variable angular velocity vector is set to be $\Omega(m_1, m_2, 1 + m_3)$; the subscript 1 and 2 denote the Greenwich meridian and 90 deg. east longitude, respectively, and the i $(= \sqrt{-1})$ is an imaginary unit. The RHS of eq. (1) represents the excitation, which is defined as $\chi_1 + i\chi_2$ [e.g., Munk and MacDonald, 1960; Lambeck, 1980]. The resonant frequency $\tilde{\sigma}_{CW}$ is defined in terms of Chandler period P and Q–value,

$$\tilde{\sigma}_{CW} \equiv \frac{2\pi}{P}(1 + i\frac{1}{2Q}), \quad (2)$$

where another imaginary unit i appears.

In view of the eqs. (1) and (2), the imaginary unit i in $\tilde{\mathbf{m}}$ as well as in $\tilde{\chi}$ is obviously for a numerical convenience that allows us to contract two variables into one variable. Even if we exchange the "real" and "imaginary" part in $\tilde{\mathbf{m}}$ and $\tilde{\chi}$, the physics of polar motion will never be changed if they are commuted at the same time. Obviously, however, we are never allowed to simply exchange the "real" and "imaginary" part of $\tilde{\sigma}_{CW}$, because they have a physical meaning of both resonant period and damping time. This apparently two different usage of imaginary unit in one governing equation seems at odds (at least for us), and motivates us to re-examine the governing equation. We consider that the complex variable representation in the time domain formulation should be consistently employed only for the purpose of numerical convenience.

The aim of this paper is two folds. First purpose is to review the damping term associated with the eq. (1), and show that it is proportional to the first order time derivative of wobble $\tilde{\mathbf{m}}$. Second purpose is to show that, when a damping term is proportional to the wobble itself, we have to multiply a correction factor to the conventional polar motion equation eq. (1).

2 Two kinds of Damping term

2.1 "Conventional" Damping Term

Decomposing the eq. (1) into real and imaginary part, we see that the damping term is shown as the second term in the LHS of eqs. (3–4) below;

$$\text{Real} \quad \frac{1}{\sigma^*}(-\frac{dm_2}{dt} + \frac{1}{2Q}\frac{dm_1}{dt}) + m_1 = 0 \quad (3)$$

$$\text{Imag} \quad \frac{1}{\sigma^*}(\frac{dm_1}{dt} + \frac{1}{2Q}\frac{dm_2}{dt}) + m_2 = 0 \quad (4)$$

Also, the eigenfrequency σ^* is equal to $\sigma_{CW}(1 + 1/4Q^2)$, and is different than the $\sigma_{CW} = 2\pi/P$. The σ^* looks unphysical, because the presence of damping Q *increases* the eigenfrequency. Nonetheless, this equation certainly allows the following damped oscillation of eqs. (5–6):

$$m_1 = \exp(-\pi t/QP)\cos(2\pi t/P), \quad (5)$$
$$m_2 = \exp(-\pi t/QP)\sin(2\pi t/P), \quad (6)$$

which is a fundamental assumption in the numerous previous studies of polar motion.

The damping term in the conventional equation is proportional to the time derivative of wobble amplitude; Munk & MacDonald (1960) states that the term above corresponds to a frictional torque acting in a direction opposite to the motion of the shell shifting in response to the wobble. We should keep in mind that, as long as one uses the conventional form of (1), the damping process for the Q in eq. (1) originates in the above frictional torque. This type of torque has been considered to work in the core [Munk & MacDonald, 1960; Lambeck, 1980].

2.2 Another choice of Damping Term

We have another simple choice for the damping term, namely, the term being simply proportional to the wobble amplitude. I regard this as a well-known 'Newtonian' damping, since the first time-derivative of the m_i corresponds to the acceleration, and the m_i itself corresponds to the velocity.

It is clearly inevitable to modify the conventional eq. (1), and we start the derivation from the scratch. For simplicity, we ignore the effect of pole tide loading upon the Chandler period, which does not affect our main concluion. The 'Newtonian' damping is the third term of LHS of the equations below,

$$A_m \frac{dm_1}{dt} + \Omega(C-A)(1-\frac{k_2^W}{k_0})m_2 + \alpha m_1 = L_2, \quad (7)$$
$$A_m \frac{dm_2}{dt} - \Omega(C-A)(1-\frac{k_2^W}{k_0})m_1 + \alpha m_2 = -L_1, \quad (8)$$

where C and A are the principal moments of inertia around the polar and equatorial axis, respectively, and $L_j (j = 1, 2)$ is the torque around the axis denoted as j; A_m represents the equatorial principal moment of inertia for only the mantle.

It should be noted that the wobble Love number k_2^W is not a complex but a real value. Lambeck (1980) used a complex Love number in his derivation of polar motion equaion. The complex Love number is, however, originally defined in a frequency domain, and might be utilized for purely harmonic phenomenon. Thus using a complex Love number in a time domain polar motion equation would be a mix of time domain and frequency domain formulation.

Combining the eqs. (7) and (8) in terms of i, we arrive at the following equation:

$$\frac{i}{\sigma_{CW}} \frac{d\tilde{\mathbf{m}}}{dt} + \tilde{\mathbf{m}}[1 + i\frac{1}{2Q^*}] = \tilde{\chi}^*, \quad (9)$$

$$\text{with} \quad \sigma_{CW} = \frac{(C-A)\Omega}{A_m}(1-\frac{k_2^W}{k_0}), \quad (10)$$
$$Q^{*-1} = \frac{2\alpha}{\sigma_{CW}}. \quad (11)$$

Here, we attached the asterisk in both the excitation and Q in order to distinguish it from the one used in eq. (1). The real and imaginary part of eq. (9) are the following;

$$\frac{1}{\sigma_{CW}}\frac{dm_1}{dt} + m_2 + \frac{m_1}{2Q^*} = \chi_2^*, \quad (12)$$
$$\frac{1}{\sigma_{CW}}\frac{dm_2}{dt} - m_1 + \frac{m_2}{2Q^*} = -\chi_1^*. \quad (13)$$

Readers can easily verify that this again comforms to the damped free oscillation of eqs. (5–6).

3 Summary and Discussion

We re-examined the polar motion equation from the viewpoint that the imaginary unit i should only be used for a numerical convenience. As long as we assume a simple linear damped oscillation of eqs. (5–6), there are two kinds of permissible damping terms, i.e., the second terms in eqs. (3–4) and the third terms in eqs. (12–13). Thus, both eqs. (3–4) and (12–13) can be the polar motion equation, while eqs. (3–4) have been conventionally employed. However, when we resort to the 'Newtonian' damping, the eq. (9) have to be employed in place of eq. (1).

Three mechanisms have been considered as candidates for the Chandler wobble damping, fluid outer core, mantle and ocean. What damping term is relevant for each candidate? While the conventional damping term is appropriate for the processes in the outer core, the viscosity of the outer core is too low to damp out the Chandler wobble [Munk and MacDonald, 1960; Lambeck, 1980]. Thus we see that the conventional damping term is geophysically implausible! On the other hand, the "Newtonian" damping is approapriate for the phase lag associated with mantle anelasticity and non-equilibrium passive pole tide.

In previous literature on the excitation of Earth's wobble, the deconvolution of observed wobble into its excitation has been carried out by way of eq. (1) by assuming the Chandler period and Q–value, and the deconvolved excitation with the eq. (1) was directly compared with geophysical excitations. If the damping term proportional to $\tilde{\mathbf{m}}$ is preferred, one cannot simply follow this procedure. Instead, one should multiply $(1 + i\frac{1}{2Q})$ to the eq. (1), so that one can evaluate the $\tilde{\chi}^*$ in eq. (9); as a caveat, one should consistently use the same Q-value both in this factor and in the $\tilde{\sigma}_{CW}$. After that, one can compare the inferred excitation with geophysical excitations such as the atmospheric/oceanic/hydrological angular momemtum.

Since the Q-value for the Chandler wobble is still not well-constrained, this has some numerical implications for the studies of polar motion excitation. We can see how the $\tilde{\chi}^*$ in eq. (9) differs from the $\tilde{\chi}$ in eq. (1), by equating the $\tilde{\chi}^*$ with the $\tilde{\chi}$ multiplied by $(1 + i\frac{1}{2Q^*})$. The mathematical relation between these two "physically distinct" excitations are:

$$\chi_1{}^* = \chi_1 - \frac{1}{2Q^*}\chi_2, \tag{14}$$

$$\chi_2{}^* = \chi_2 + \frac{1}{2Q^*}\chi_1. \tag{15}$$

If the Q–value is as high as O(100) or greater as estimated in most previous studies, and if both χ_1 and χ_2 have similar amplitudes, the numerical bias would be at most a few percent in spite of an appreciable difference in the appearance between eqs. (3–4) and (12–13). This bias would be proportionally larger if the Q is actually smaller, for example, at about 50 as determined by Furuya & Chao (1996). Moreover, when either χ_1 or χ_2 has greater ampltiude than the other, as observed in the annual atmospheric angular momentum (Chao and Au, 1991), the estimated excitation can be appreciably different.

Although the imaginary unit i is often used in an eigen frequency in classical mechanics to account for a damping, we need to be careful in dealing with a damping of polar motion; same thing would be true for a nearly diurnal free wobble. From the view point that the imaginary unit i should be used just for numerical convenience, the two is in eq. (1) are inconsistent on a physical front. The damping term in eq. (1) concerns about processes in the fluid outer core, which is surely negligible. The physical inconsistency is leading to a numerical bias as seen in eqs. (14–15). We need to keep in mind that the bias will introduce significant biases in the estimation of the Chandler period and Q itself.

4 Acknowledgement

This study was motivated from discussions with Dr. Ben Chao when the author visited GSFC/NASA in 1995. The author expresses sincere thanks to him, who also carefully read the original manuscript. The author also thanks Dr. John Wahr for discussion.

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